Multiplicity in Cascade Transmission Line Synthesis—Part I

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Abstract—The synthesis of a stepped equal-length transmission line structure to a given insertion loss function sometimes leads to a multiplicity of untrivially related solutions. The philosophy of synthesis is explored to understand this lack of uniqueness and a new statement of sufficiency and necessity of synthesis is developed based only on an insertion loss statement. The conditions are developed for nonuniqueness to occur and it is observed that these conditions are particularly prevalent in transmission line couplers. It is precisely the nonuniqueness of couplers that accounts for both the asymmetric and symmetric realizations. The symmetric coupler is generally the more difficult of the two to design and both exact and approximate methods of design are given in this case.

I. Introduction

TRANSMISSION line synthesis characteristically associated with quarter-wave transformers and band-pass filters [1], [2] has recently been extended to coaxial couplers [3]–[5]. In independent studies of the coupler synthesis problem it became clear to the authors that the bridge between asymmetric and symmetric couplers lay in the nonuniqueness of the synthesis of a transfer function by means of an equal section length transmission line cascade.

This paper has two major purposes. Its first intent is to explore the source of nonuniqueness and to provide examples of multiple synthesis. In exploring this source we shall seek out those criteria which lead to classes of structures possessing unique synthesis. The second major purpose of this paper is the further exploration of realizability criteria towards the end of demonstrating realizability of all the solutions of synthesis. This investigation leads to the following simple statement of realizability in terms of cascade transmission line synthesis:

Let θ be the electrical length of a constant relative impedance transmission line where all real positive characteristic impedances are realizable. Then, any even polynomial loss function L (i sin θ) may be realized as a cascade of transmission lines of equal section length θ to within an added terminating transformer if, and only if, L (i sin θ) ≥ 1 for all real values of θ. The terminating transformer vanishes when $L(O)=1$.

The results of the theory show that most of the filter and quarter-wave transformer designs fall into the class of structures having unique synthesis. This class, we find, is characteristic of networks having optimal transmission properties over a band. Couplers, on the other hand, are designed to produce a wide band of constant reflection. It is this feature which produces multiplicity and accounts for the evolution of the two coupler types; the symmetric and asymmetric structures. Where flatness of coupling is the major concern, it is shown that the asymmetric coupler has superior features to those of the symmetric device. The flat 90° differential phase of the coupled and transmission ports of the symmetric coupler give it unique advantages, however, and may be the dominant feature of design.

The paper has ten sections divided as follows: Section II demonstrates synthesis of any even loss polynomial in sin θ meeting the restrictions of the statement of realizability. Section III proves realizability by a root locus procedure and an equivalence to positive real function theory is given in Section IV. Section V explores multiple synthesis and provides criteria for uniqueness. Section VI examines the synthesis problem of couplers and shows the general existence of multiple solutions for three- or more section structures. It shows qualitative reasons why asymmetric structures are flatter in coupling than are symmetric structures. Section VII explores two variants of four-section asymmetric design and compares differential phase of the two qualitatively and quantitatively. Section VIII deals with the three-section symmetric coupler and shows that there exists a three-section asymmetric coupler with an identical transfer function. Both an approximate and an exact design procedure are given. An approximate design is also indicated for a five-section symmetric coupler. Section IX takes a concluding look at two aspects of transmission line structures. The first relates to the equivalence of filters and quarter-wave transformers using the realizability statement, and the second is concerned with the applicability of asymmetric couplers to mixers. Section X contains brief comments about the literature with respect to some aspects of the present paper.

The paper will be presented in two parts, the first containing Sections I through V. The intent is to permit two self-contained portions; the first dealing, essentially, with general analysis and the second providing more specific results.

II. Synthesis Procedure

The synthesis procedure, as well as the demonstration of realizability, is an extension of that employed in [2]. In that paper an insertion loss function L was chosen to be of the form $1+R_n^2(\sin\theta)$, where R_n is an even or odd polynomial of degree n. Here we shall consider the

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more general problem where $L(i \sin \theta)$ is any even polynomial function of $\sin \theta$ subject only to the condition $L \geq 1$ for all real values of θ. The procedure bears resemblance to a Darlington [6] synthesis but is modified by the use of radical factors as introduced in [2].

Let us introduce the complex variable $p = i \sin \theta$ which operates for distributed structures in a manner reminiscent of the fashion in which $p = i\omega$ operates for lumped elements. The insertion loss function $L(p)$ is related to the reflection factor $k(p)$ as follows:

$$k(p)k(-p) = \frac{L(p) - 1}{L(p)}. \tag{1}$$

The synthesis method requires that the right-hand side of (1) be factored into two functions, the first being a real function of $+p$ and the second being the same function of $-p$.

A simple means of identifying the factors of (1) is accomplished by a root sorting process. Let p_i be a root of either numerator or denominator. If p_i is real, then the evenness of $L(p)$ requires the existence of the negative real root. The sorting process requires that p_i be associated with one factor and that $-p_i$ be associated with the other. If p_i is complex the demand that $k(p)$ be real requires a segregation of both p_i and p_i^* into one of the factors with the residual roots $-p_i$ and $-p_i^*$ going into the other factor. Equation (1) then decomposes into

$$k(p)k(-p) = \frac{a^2 \prod_i (p - p_i) \prod_i (-p - p_i)}{a^2 \prod_j (p - p_j) \prod_j (-p - p_j)} \tag{2}$$

where a^2 is a real numeric factor, and where the products in both numerator and denominator range over all permissible independent values of p_i and p_j together with their complex conjugates. Recognizing the identity

$$\begin{aligned}(p - p_i)(p + p_i) &= (1 + p_i^2)\left(p - \sqrt{1 + p^2}\,\frac{p_i}{\sqrt{1 + p_i^2}}\right) \\ &\quad \cdot \left(p + \sqrt{1 + p^2}\,\frac{p_i}{\sqrt{1 + p_i^2}}\right)\end{aligned} \tag{3}$$

we may make the following identification of $k(p)$ in (2):

$$k(p) = \frac{\pm a \prod_i \sqrt{1 + p_i^2}\left(p - \sqrt{1 + p^2}\,\frac{p_i}{\sqrt{1 + p_i^2}}\right)}{a \prod_j \sqrt{1 + p_j^2}\left(p - \sqrt{1 + p^2}\,\frac{p_j}{\sqrt{1 + p_j^2}}\right)}. \tag{4}$$

The motivation in factoring (3) as we do stems from the desire to produce polynomial functions in terms of both sines and cosines of some transmission line length. If $p = i \sin \theta$, then $\sqrt{1+p^2} = \cos \theta$ and (4) will be shown to represent the reflection of a transmission line cascade. Network stability [7] requires that $k(p)$ have only left-half plane poles in the variable $p/\sqrt{1+p^2}$ so that all $p_j/\sqrt{1+p_j^2}$ in (4) have negative real parts. There is no similar restriction on the zeros of $k(p)$ so that the choice of p_i in (4) is unrestricted. It is precisely here in this choice that transmission line synthesis may be nonunique.[1]

If $L(p)$ is a polynomial of degree $2n$ then, from (4), $k(p)$ has the following form:

$$k(p) = \frac{M_n(p) + \sqrt{1 + p^2}\, N_{n-1}(p)}{P_n(p) + \sqrt{1 + p^2}\, Q_{n-1}(p)} \tag{5}$$

where it is assumed for the moment that M, N, P, and Q are real even or odd polynomials of degree corresponding to subscript notation. Since

$$k(-p) = \frac{(-1)^n(M_n(p) - \sqrt{1 + p^2}\, N_{n-1}(p))}{(-1)^n(P_n(p) - \sqrt{1 + p^2}\, Q_{n-1}(p))} \tag{6}$$

we have from (1), (5), and (6)

$$\begin{aligned}k(p)k(-p) &= \frac{(-1)^n(M_n^2(p) + (1 + p^2)N_{n-1}^2(p))}{(-1)^n(P_n^2(p) - (1 + p^2)Q_{n-1}^2(p))} \\ &= \frac{L(p) - 1}{L(p)}.\end{aligned} \tag{7}$$

Equation (7) produces the result

$$M_n^2(p) - (1 + p^2)N_{n-1}^2(p) = (-1)^n(L(p)) - 1) \tag{8}$$

$$P_n^2(p) - (1 + p^2)Q_{n-1}^2(p) = (-1)^n L(p). \tag{9}$$

Equation (5) may be made to yield the impedance function of the terminated four pole through the relationship $Z = 1 + k/1 - k$ and we find, from (5),

$$Z(p) = \frac{(P_n + M_n) + \sqrt{1 + p^2}\,(Q_{n-1} + N_{n-1})}{\sqrt{1 + p^2}\,(Q_{n-1} - N_{n-1}) + (P_n - M_n)}. \tag{10}$$

Given a transfer matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for a four pole it is well known that the input impedance to the four pole terminated by a unit load is given by $Z = A + B/C + D$. One is then led easily to the tentative recognition of the transfer matrix to be either of the form

$$T_n = \begin{bmatrix} (P_n + M_n) & \sqrt{1+p^2}\,(Q_{n-1} + N_{n-1}) \\ \sqrt{1+p^2}\,(Q_{n-1} - N_{n-1}) & (P_n - M_n) \end{bmatrix} \tag{11}$$

or of the form

$$T_n = \begin{bmatrix} \sqrt{1+p^2}\,(Q_{n-1} + N_{n-1}) & (P_n + M_n) \\ (P_n - M_n) & \sqrt{1+p^2}\,(Q_{n-1} - N_{n-1}) \end{bmatrix}. \tag{12}$$

[1] These features result directly from a root locus procedure. Actually, no direct appeal need be made to PRF theory, but compatibility with it is established in Section IV.

The determinant of T_n in (11) is

$$(P_n^2 - M_n^2) - (1 + p^2)(Q_{n-1}^2 - N_{n-1}^2) = (-1)^n \quad (13)$$

where the equality stems from (8) and (9) and the determinant of T_n in (12) is $(-1)^{n+1}$. Since the determinant of the transfer function of a reciprocal four pole is unity, (11) is to be identified with a structure having an even number of elements while (12) corresponds to an odd element structure. This identification is also apparent from the requirement that major diagonal terms be real and minor diagonal terms imaginary.

Equations (11) and (12) do indeed possess the necessary form for n-element transmission line structures. Recognizing that a single transmission line of characteristic impedance Z possesses the transfer matrix

$$\begin{bmatrix} \sqrt{1+p^2} & Zp \\ \frac{p}{Z} & \sqrt{1+p^2} \end{bmatrix} \quad (14)$$

it is evident that (11) and (12) go into one another via matrix multiplication with (14), corresponding to the addition of another section to the four pole. Since (12) and (14) correspond for $n=1$, proof by induction is complete that (11) and (12) have the proper form to represent an equal-length transmission line cascade.

Given an appropriate insertion loss function we shall now demonstrate the synthesis method. We arbitrarily take n odd and expand (12).

$$T_n = \begin{bmatrix} \sqrt{1+p^2}(a_{n-1}p^{n-1} + a_{n-3}p^{n-3} + \cdots) & b_np^n + b_{n-2}p^{n-2} + \cdots \\ c_np^n + c_{n-2}p^{n-2} + \cdots & \sqrt{1+p^2}(d_{n-1}p^{n-1} + d_{n-3}p^{n-3} + \cdots) \end{bmatrix}. \quad (15)$$

Assuming $n>1$ the leading term of the determinant of (15) is zero since the determinant is identically unity. Hence

$$a_{n-1}d_{n-1} - b_nc_n = 0, \qquad n > 1. \quad (16)$$

If (15) implicitly contains n transmission lines, we may remove one by appending a line length $-\theta$ on the left having some characteristic impedance Z. Performing this operation on (15) we have

$$T_{n-1} = \begin{bmatrix} (a_{n-1} - Zc_n)p^{n+1} + O(p^{n+1}) & \sqrt{1+p^2}[(b_n - Zd_{n-1})p^n + O(p^{n-2})] \\ \sqrt{1+p^2}\left[\left(c_n - \frac{a_{n-1}}{Z}\right)p^n + O(p^{n-2})\right] & \left(d_{n-1} - \frac{b_n}{Z}\right)p^{n+1} + O(p^{n-1}) \end{bmatrix}. \quad (17)$$

In view of (16) a choice of

$$Z = \frac{a_{n-1}}{c_n} = \frac{b_n}{d_{n-1}} \quad (18)$$

provides for the reduction of the respective components of T_{n-1} to polynomials of one degree less than those for T_n.

We may continue the procedure of removing transmission lines until $n=0$ for which we obtain

$$T_0 = \begin{pmatrix} a_0 & 0 \\ 0 & d_0 \end{pmatrix} \quad (19)$$

and which clearly represents a transformer since $a_0d_0 = 1$. If as a special assumption we require that $L(0)=1$, namely that there is no reflection when the transmission lines are of zero length, then $a_0 = d_0$ and synthesis may be accomplished by transmission lines only.

III. Realizability

It was assumed, in relation to (5), that M_n, N_{n-1}, P_n, and Q_{n-1} were all real polynomials and that there had been no cancellation of common factors between numerator and denominator. Because it is necessary that the various matrix terms of (11) and (12) be real in p, we are obliged to show that this assumption is, indeed, correct. To do this we shall now state and prove two properties of the loss function under the two assumptions that $L(p) \geq 1$ for $|p| \leq 1$ for p on the imaginary axis and that $L(p)$ be an even polynomial. These assumptions follow, respectively, from conservation of energy and reciprocity.

A) The number of real roots[2] of $L(p)$ and $L(p)-1$ differ by a multiple of four.

B) The leading coefficient of $L(p)$ has the sign $(-1)^{n_r/2}$ where n_r is the number of real roots of either $L(p)$ or $L(p)-1$.

To prove property A consider the loss function $L(p)-\alpha$, where α is a constant which ranges in value from zero to a maximum of one. For $\alpha<1$ there are no roots on the imaginary axis for $|p|<1$ by assumption. There are two possible ways in which roots may leave the real axis as α varies:

a) The roots on the real axis merge in pairs and then enter into the complex plane to form quartets of complex roots (i.e., p_i, p_i^*, $-p_i$, $-p_i^*$).

b) The roots pass from the real to the imaginary axis through the point at infinity.

This last mechanism, however, is absurd since a root passing through the point at infinity implies a vanishing of the leading coefficient of $L(p)-\alpha$ as α varies.

[2] Roots at zero are considered real roots in all subsequent discussions.

The first mechanism of real root loss is, therefore, the only permissible one for α less than, and arbitrarily close to, unity. For $\alpha \equiv 1$, even clusters of roots may appear on the real axis at the origin since roots of $L(p)-1$ are permitted on the imaginary axis for $|p|<1$. However, this does not constitute a real root loss. Since the only mechanism of real root loss is through complex root quartets, property A is proven.

Property B is also simply demonstrated. Since $L(p)$ is an even polynomial it has the representation

$$L(p) = \prod_{i,j,k} K(p^2 - a_i^2)(p^2 + b_j^2)(p^2 - c_k^2)(p^2 - c_k^{*2})$$

where a_i are the real roots, ib_j the imaginary roots, and c_k the complex roots. As a particular case of the assumptions on $L(p)$, $L(0)>0$. This requires that K have the sign $(-1)^{n_r/2}$ where n_r is the number of real roots.

We may now prove the desired result that the various polynomials in (5) are all real. In considering either numerator or denominator in (4), there are no complex terms introduced by a complex root p_i since there is always present another root p_i^* to restore reality. In the particular case of the numerator, roots along the imaginary axis for $|p|<1$ are formed from a double set because of the non-negative nature of $L-1$ for that region. It is thus possible in sorting out the respective roots of $k(p)$ and $k(-p)$ to assign one such imaginary root and its complex conjugate to one of the factors and the other pair of roots to the other factor.

Equation (4) becomes imaginary with the introduction of those imaginary roots in either numerator or denominator for which $|p| \geq 1$. Since these roots are, in general, simple there is a root assignment of one imaginary root to $k(p)$ and its negative to $k(-p)$. These roots are now unpaired and lead to imaginary coefficients in (4) since $\sqrt{1+p_i^2}$ is imaginary. The complete coefficient is real or imaginary in relation to the number of pairs of imaginary roots for which $|p| \geq 1$. In particular either numerator or denominator of (4) is, respectively, real or imaginary as is the quantity

$$a i^{(n_i - n_i')/2}$$

where n_i is equal to the total number of imaginary roots and where n_i' is the number of imaginary roots for $|p|<1$. Recalling that imaginary roots $|p|<1$ are double and are paired with their complex conjugates, n_i' is a multiple of four. Further, from (2) and properties A and B, the leading coefficient of either numerator or denominator has the same sign as

$$\operatorname{sgn} (-1)^n a^2 = (-1)^{n_r/2}$$

where, unambiguously, n_r is the number of real roots of either numerator or denominator. Taking all the above into account, either numerator or denominator is real or imaginary as is the quantity

$$(-1)^{(2n+n_r+n_i)/4}.$$

Since $n_r+n_i=2n$ (mod 4) we obtain the final result that both numerator and denominator are real. Hence, M_n, N_{n-1}, P_n, and Q_{n-1} are all real polynomials.

Applying the matrix reduction technique associated with (17) one may now find an array of characteristic impedances which correspond to the insertion loss function within a transformer. Since all of these impedances are real, the demonstration of realizability becomes that of showing that all the impedances are positive. We shall employ a root locus procedure[3] to show realizability similar to that employed in [2].

The root locus procedure starts with the root locations of a structure which is known to exist. Always maintaining the fundamental condition that $L(p)-1 \geq 0$ for imaginary roots $|p| \leq 1$, the roots of $L(p)-1$ are varied continuously from that array corresponding to the existing structure to that array corresponding to the loss function to be synthesized. If a continuous locus exists and if the structure is continuously realizable along the root locus, then the synthesis is realizable.

Certainly an n section structure exists giving us an admissible array of roots corresponding to some $L(p)-1$. Let us now consider root loci of $L(p)-1$ which are restricted as follows:

1) All roots maintain reflection symmetry about both real and imaginary axes.
2) Roots traveling on the imaginary axis for $|p|<1$ must remain paired (double).
3) Roots on the imaginary axis for $|p| \geq 1$, or real axis roots, may be simple.
4) Roots of any multiplicity may be interchanged between the real and imaginary axes through the point at infinity.

Through the preceding rules one always maintains the evenness of the polynomial $L(p)$ together with a single sign of $L(p)-1$ on the imaginary axis for $|p| \leq 1$. It is also evident that every consistent root pattern of $L(p)-1$ is permitted by the foregoing loci.

It is implicit in the preceding that the sign of the leading coefficients of $L(p)-1$ will always be maintained equal to $(-1)^{n_r/2}$. There is no discontinuity in changing this sign as roots pass through infinity since such roots imply a vanishing of the leading coefficient of $L(p)$ to a degree consistent with the multiplicity of roots having passage. Hence $L(p)-1$ remains non-negative on the imaginary axis for $|p|<1$ for all possible loci.

We have shown that admissible loci exist maintaining both the even and non-negative character of $L(p)-1$ which carry us from the root array of a physically existing structure to the root array for a specific $L(p)-1$. We may, therefore, continuously construct an array of real characteristic impedances according to (17) where,

[3] Root locus procedures find frequent application in control theory. See, for example, J. G. Truxal, *Automatic Feedback Control System Synthesis*. New York: McGraw-Hill, 1955.

at least in the neighborhood of the initial points of the loci, every impedance is positive. If one or more impedances change sign along the root loci, each must first pass through a value of zero or infinity. It is easily shown that no multiplicity exists of zero or infinite characteristic impedance sections which may produce other than an infinite insertion loss for finite values of p. Since this contradicts the finite polynomial nature of $L(p)$, there may be no sign changes, and the network, is realizable.

We have now established the sufficiency of the synthesis procedure and it remains for us to demonstrate that this procedure exhausts all possible syntheses. We have shown in Section II by an inductive proof that (11) and (12) represent the necessary matrix forms for an equal line length structure. Equation (13) describing a unit determinant condition is necessary from considerations of reciprocity. Equation (9) describing the loss function is a direct consequence of (11) or (12), and (8) derives directly from (9) and (13).

Either (8) or (9) may be written in the form

$$(A_n(p) + \sqrt{1+p^2}\, B_{n-1}(p)) \cdot (A_n(p) - \sqrt{1+p^2}\, B_{n-1}(p)) = F_{2n}(p)$$

where F_{2n} is an even polynomial of degree $2n$. Transform to the variable $\zeta = \sqrt{(1+p^2)}/p$ so that we have a new equation of the form

$$(\alpha_n(\zeta) + \beta_{n-1}(\zeta))(\alpha_n(\zeta) - \beta_{n-1}(\zeta)) = \phi_{2n}(\zeta)$$

where α_n, β_{n-1}, and ϕ_{2n} are appropriate odd or even polynomials, respectively. Since the linear factors of ϕ_{2n} are unique, only a finite number of organizations of n factor pairs are permissible compatible with the form of the left-hand side of the equation. The left-hand side is the product of complex conjugate factors of degree n and synthesis of (11) or (12) requires that just one of these factors be chosen. The synthesis procedure employed exhausts all permissible arrangements of the n linear factors, and no other synthesis is possible.

IV. Connection With PRF Theory

A root locus demonstration of realizability was employed in the Section III because it tied the examination of realizability to the loss function itself and because it meaningfully retained the variable $p = i \sin \theta$ in which the problem is specified. One is then led to direct statements about the admissibility of the loss function itself, namely the function specified for synthesis, as opposed to statements relating to impedance functions in a new variable $\zeta = -i \cot \theta$ as is required by Richards' theorem [7] and the use of positive real function theory.

We shall show, nevertheless, that the root locus demonstrations employed are consistent with PRF theory. It is to be recalled that the root locus method required some initial structure to establish an initial root array for $L(p) - 1$. Let us start with an n section equal impedance cascade structure which has the transfer matrix

$$T = \begin{bmatrix} \cos n\theta & iZ \sin n\theta \\ \dfrac{i}{Z} \sin n\theta & \cos n\theta \end{bmatrix} \tag{20}$$

and which, with a unit termination, produces the reflection factor

$$k = \frac{\dfrac{i}{2}\left(Z - \dfrac{1}{Z}\right) \sin n\theta}{\cos n\theta + \dfrac{i}{2}\left(Z + \dfrac{1}{Z}\right) \sin n\theta}. \tag{21}$$

Let us consider the poles of k which correspond to the choice of appropriate roots of $L(p)$ in (4). The poles θ_m are found from the equation

$$\epsilon^{i2n\theta_m} = \left[\frac{Z + \dfrac{1}{Z} - 2}{Z + \dfrac{1}{Z} + 2}\right]. \tag{22}$$

From (22) we obtain

$$\zeta_m = -i \cot \theta_m = \frac{\epsilon^{i2\pi m/n} \left[\dfrac{Z + \frac{1}{Z} - 2}{Z + \frac{1}{Z} + 2}\right]^{1/n} + 1}{\epsilon^{i2\pi m/n} \left[\dfrac{Z + \frac{1}{Z} - 2}{Z + \frac{1}{Z} + 2}\right]^{1/n} - 1} \tag{23}$$

where m is an integer. From (23)

$$\operatorname{Re}(\zeta_m) = \frac{\left[\dfrac{Z + \frac{1}{Z} - 2}{Z + \frac{1}{Z} + 2}\right]^{2/n} - 1}{\left| \epsilon^{i2\pi m/n} \left[\dfrac{Z + \frac{1}{Z} - 2}{Z + \frac{1}{Z} + 2}\right]^{1/n} - 1 \right|^2} < 0 \tag{24}$$

so that all poles of k are in the left hand of the ζ plane.

In terms of the variable p the quantity ζ is given as

$$\zeta = \frac{\sqrt{1+p^2}}{p}. \tag{25}$$

ζ is imaginary for that range in which p is imaginary and $|p| \leq 1$. Since this is exactly the range for which $L(p) \geq 1$, there may be no roots of $L(p)$ and, consequently, no poles of k on the imaginary ζ axis as the roots of $L(p)-1$ trace their respective paths. Since the point at infinity of ζ corresponds to $p=0$ where, again, there is no root of $L(p)$, k may have no poles in the ζ plane approaching infinity. The poles of k in the ζ plane which were initially all in the left-half plane are trapped in the left-half plane since they may not leave either through the imaginary axis or the point at infinity.

The imaginary ζ axis, as indicated, corresponds to the range $\infty > L \geq 1$ so that

$$|k|^2 = \frac{L-1}{L} < 1. \tag{26}$$

Since k is analytic in the right-half plane and has a magnitude less than 1 on the imaginary axis, it has a magnitude less than 1 everywhere in the right-half plane. Thus $Z=(1+k)/(1-k)$ is a positive real function for all admissible root locations of $L-1$.

Inverting (25) we find $p=1/\sqrt{\zeta^2-1}$. Substituting in (10) and multiplying numerator and denominator by $(\zeta^2-1)^{n/2}$, we find

$$Z(\zeta) = \frac{(\zeta^2-1)^{n/2}(P_n+M_n)+\zeta(\zeta^2-1)^{(n-1)/2}(Q_{n-1}+N_{n-1})}{\zeta(\zeta^2-1)^{(n-1)/2}(Q_{n-1}-N_{n-1})+(\zeta^2-1)^{n/2}(P_n-M_n)}. \tag{27}$$

Letting $m_{1,2}$ stand for a real even or odd polynomial in ζ and $n_{1,2}$ stand for an odd or even polynomial, (27) has the form

$$Z(\zeta) = \frac{m_1+n_1}{m_2+n_2}.$$

From the unity determinant condition of (13)

$$m_1m_2 - n_1n_2 = (-1)^n C^2(\zeta^2-1)^n \tag{28}$$

where C is any real number multiplying numerator and denominator of (27).

$Z(\zeta)$ is, therefore, positive real and, by (28) also meets the second of Riblet's two conditions for cascade transmission line realizability [1]. Since every root array corresponding to the loci of $L(p)-1$ leads to a realizable structure, the endpoints of the loci are also realizable, completing the statement of realizability.

V. Multiplicity in Loss Function Synthesis

Network determination corresponding to a given loss function depends on the specific choice of half the number of roots of $L(p)-1$ to form the reflection function $k(p)$. The roots chosen obey the following selection conditions.

1) With the exception of imaginary roots for which $|p| \geq 1$, the roots of $k(p)$ have reflection symmetry about the real axis.
2) With the exception of imaginary roots for which $|p| > 1$, the roots of $k(p)$ and $k(-p)$ have a mutual reflection symmetry about the imaginary axis.
3) For every imaginary root in $k(p)$ a complex conjugate root is to be found in $k(-p)$.

The selection rules do not enforce a unique choice by any means since one may take any root p_i, and its conjugate if needed, and exchange it with its negative $-p_i$. Root locus considerations as employed earlier show that realizability is not compromised and the loss function remains invariant.

Since nonuniqueness is the rule and not the exception we should like to seek the cases in which root inversion, namely the transformation of p_i to $-p_i$, leads either to trivial change, or to no change at all, in structural synthesis. In a first examination all the roots of $k(p)$ are inverted and we seek to determine the effect. In a second study we shall attempt to determine under what circumstances a network is insensitive to the inversion of any number of the roots of $k(p)$.

A. Complete Root Pattern Inversion

While it might be supposed that the transformation of a root pattern of $k(p)$ to one having an inverted pattern might lead to trivial results, this is not necessarily the case. Let us consider the transformation in which all roots of $k(p)$, p_i, transform into their negatives. The numerator of (5) becomes

$$(-1)^n(M_n(-p) + \sqrt{1+p^2}\,N_{n-1}(-p)) = M_n(p) - \sqrt{1+p^2}\,N_{n-1}(p)$$

and is characterized by the transformation $N_{n-1}(p) \rightarrow -N_{n-1}(p)$. Equation (12) shows that a change of sign of $N_{n-1}(p)$ interchanges the major diagonal terms for n odd, while the minor diagonal term of (11) interchanges for n even. As we shall show now, this shifting of matrix terms implies that total root inversion reverses the odd section structure and produces the reverse dual of the even number section structure.

Since $k(p)$ is defined by the square of its magnitude, it is indeterminate to within a minus sign. A reflection function $-k(p)$ corresponds to the dual of the structure corresponding to $k(p)$, where the dual interchanges the roles of electric and magnetic fields. If electric and magnetic fields transform into one another, then the terms of the dual transfer matrix is obtained by an interchange, respectively, across both diagonals; $A \rightleftarrows D$, $B \rightleftarrows C$. A network reversal only interchanges the major diagonals so that $A \rightleftarrows D$. A minor diagonal interchange then corresponds to the reverse of the dual.

If the original root pattern provides a given network and its dual then, irrespective of n even or odd, complete root inversion provides both the network reverse and reverse dual. Let it be assumed that the prototype network contains a terminating transformer N. If we recognize the transformation which carries a transformer from the left of an impedance array to the right of the array by multiplying each characteristic impedance by the square of the transformer value, we have the results of Fig. 1.

It is all but trivial that an array of transmission lines of a given characteristic impedance has the same insertion loss as an array having the identical values of admittances, and it is similarly trivial that a network under reversal has the same insertion loss as the forward network. In the case of $N=1$, therefore, no information of major value is contained in the inverse pattern over that of the original pattern.

The case of $N\neq 1$ is, however, of more important interest. The networks containing terminating transformers are all filters possessing the same loss characteristics. The transformer combined with the load represents a mismatched termination, and the array of transmission lines preceding the transformer represents a network providing a specified insertion loss function from the generator into a mismatched load. In viewing both the original network and the reversed dual network of Fig. 1 one observes, in general, two entirely different transmission line networks providing the same transfer function for all frequencies into the same mismatched load.

Under certain circumstances the network and its reversed dual might be identical. We now ask what these conditions are. The transfer matrix of the reverse dual differs from that of the prototype network by a reversal of minor diagonal terms. Since the two descriptions must be identical we find from (11) and (12)

$$N_{n-1} = 0; \quad n \text{ even}$$

$$M_n = 0; \quad n \text{ odd}$$

$$L = 1 + M_n^2(p); \quad n \text{ even} \tag{29a}$$

$$L = 1 + (1 + p^2)N_{n-1}^2(p); \quad n \text{ odd.} \tag{29b}$$

Equations (29a) and (29b) are phrased in polynomials M_n and N_{n-1} which are both even. Recognizing that $1+p^2=\cos^2\theta$ both (29a) and (29b) may be combined to yield as the necessary condition of network identity

$$L = 1 + R_n^2(\cos\theta) \tag{30}$$

where R_n is either an even or an odd polynomial of degree n.

It is simple to understand why (30) provides the identity of the network and its reversed dual. Since all the roots of $L-1$ are double it is possible to factor $|k(p)|^2$ into identical factors each containing all the roots of $R_n(\cos\theta)$. Since R_n is either an odd or even polynomial, for every root in a factor its negative is contained as well. Hence root inversion leaves $k(p)$ invariant to within a sign change if $k(p)$ and $k(-p)$ have identical roots.

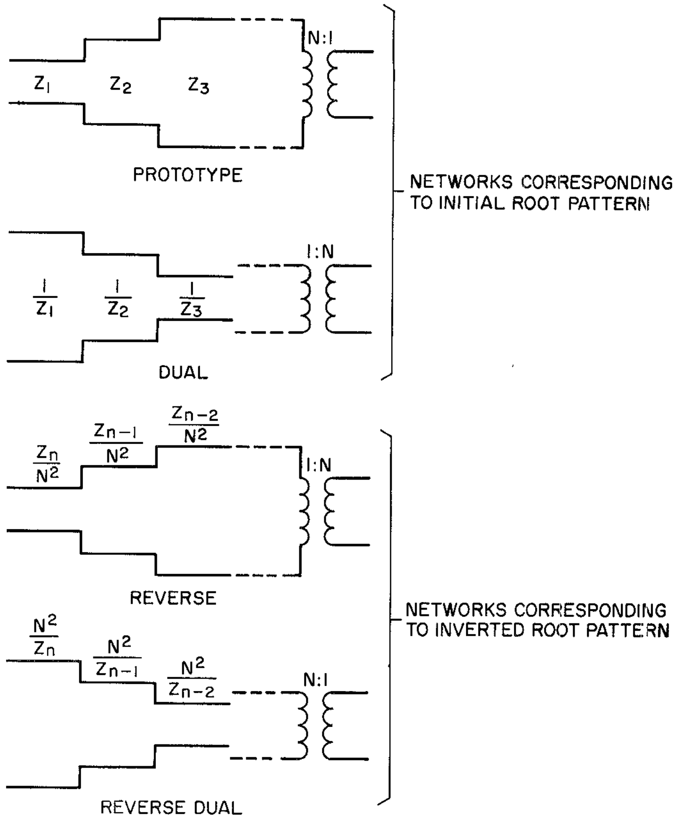

Fig. 1. Networks corresponding to initial and inverse patterns of $k(p)$.

We define a root pattern of $L-1$ to be "basic" if it is defined to within an inversion through the origin. If $L-1$ has no roots at the origin then $N\neq 1$, and we have previously shown that a basic pattern implies two structural realizations. However, these realizations merge under the restrictions that $L(p)-1$ have double roots and that $k(p)$ and $k(-p)$ be made to have identical roots. Figure 2 shows a double root pattern of $L-1$ which, for case (a), leads to identical roots for $k(p)$ and $k(-p)$ and, thus, in turn leads to unique realization under root inversion, while case (b) leads to nonunique realization.

It is of interest that past realizations only have considered loss functions for a quarter-wave transformer of precisely the form $L=1+R_n^2(\cos\theta)$ with identical root assignments to $k(p)$ and $k(-p)$. Figure 1 shows an inversion symmetry of the characteristic impedances of a quarter-wave transformer, seen in comparing the original network with its reversed dual, where

$$Z_k Z_{n-k+1} = N^2. \tag{31}$$

Equation (31) is seen to apply precisely to Fig. 10 of [2].

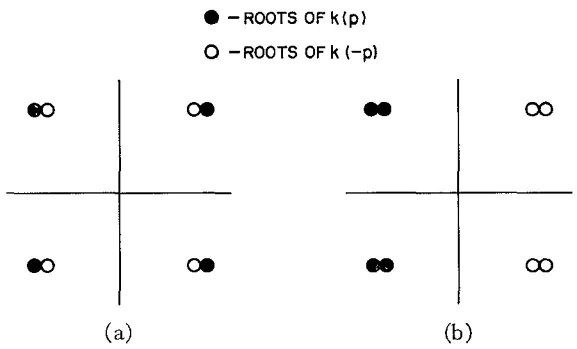

Fig. 2. Basic patterns leading to (a) unique, (b) nonunique realization.

B. Unique Synthesis

Having observed invariance for one basic pattern under root inversion, we next seek to determine under what circumstances the basic patterns themselves are identical. If the basic patterns are identical then there are, at most, two essentially different structures. Further, if the single basic pattern is invariant along the lines discussed, then there is a unique realization.

Given a basic pattern we may shift to another basic pattern by causing any one root p_i to transform to its negative. Simultaneously p_i^* transforms to its negative if p_i^* is contained as a root within the pattern. The simultaneous transformations $p_i \to -p_i$ and $p_i^* \to -p_i^*$ lead to identity only if $p_i = -p_i^*$, which implies that p_i is imaginary. Further $|p_i| < 1$ if both the root and its conjugate are to be contained in the same factor $k(p)$ for reasons given in Section III. Since $p_i = \pm i$ is the point at infinity in the transformation $p_i/\sqrt{1+p_i^2}$, a transformation of $p_i = i$ into its negative also leads to identity.

Assume for the moment that the single roots $p_i = \pm i$ are not contained in $L-1$ so that there are only double roots on the imaginary axis for $-1 < -ip < +1$. Therefore, $L-1 \sim F_n^2(p)$, where F_n is an appropriate nth degree polynomial. If the roots $p_i = \pm i$ are contained in $L-1$, then $L-1 \sim (1+p^2)F^2_{n-1}(p)$. Both of these results may be phrased alternatively as follows:

$$L = 1 + R_n^2(\sin\theta) \tag{32a}$$

$$L = 1 + R_n^2(\cos\theta) \tag{32b}$$

where R_n^2 is an even polynomial of degree $2n$ and where there are n roots of R_n contained in the interval $-(\pi/2) \leq \theta \leq (\pi/2)$.

We shall now associate (32a) with filters and (32b) with quarter-wave transformers. For n even, there is no clear distinction between (32a) and (32b) since $\sin\theta$ and $\cos\theta$ occur to an even degree and are interchangeable. The case of n odd is, however, clearly distinguishing. We define a filter structure to be composed of an array of transmission lines without a terminating ideal transformer or, at worst, with a transformer close to unity turns ratio to account for a small center-band mismatch. A filter structure is, therefore, characterized by the condition that $L=1$ for $\theta=0$, to which only (32a) applies in general for n odd. A quarter-wave transformer, on the other hand, has a terminating ideal transformer which represents the load mismatch and is characterized by a lossless transformation at $\theta=90°$. Clearly, only (32b) applies in general for n odd.

Equation (32b), by its very construction, has identical roots for $k(p)$ and $k(-p)$ and, since it falls into the form of (30), it can yield only one quarter-wave transformer construction. If we consider the mere reversal of a structure to lead to identity, then (32a) shows a unique filter construction for n even and a double realization for n odd. This result is obtained as follows. The double imaginary root structure of $L-1$, with all roots lying in the domain $|p| < 1$, leads to $N_{n-1}(p) \equiv 0$ in (4). Equation (11) shows the even section filter to be identical to its reverse dual, while (12) shows the odd section filter to be symmetric. Since the dual of the even structure leads to a simple reversal, no new structure results from dualization so that the even section filter is unique. The dual of the odd section filter is, however, a different structure, so that there is a double realization in that case.

With the exception of the Gaussian response function, the usual treatment in the literature on filters and quarter-wave transformers relates to the design of Chebyshev and Butterworth responses. The Chebyshev polynomial has n roots over the range of $-(\pi/2) \leq \theta \leq (\pi/2)$ while the Butterworth synthesis of $L-1$ concentrates all roots at the origin. From the preceding discussion we observe that these are exactly the conditions for unique synthesis and account for the failure to describe multiple realizations in the literature. Coupler design, as we shall observe in Part II, requires completely dissimilar design premises to those of the filter and quarter-wave transformers, and the nonuniqueness of synthesis is most pertinent.

C. Invariant Properties or Multiple Synthesis

We have examined situations in Sections V-A and V-B in which synthesis was insensitive to some degree to root inversion. It is of interest, however, that two invariant features persist in all multiple syntheses irrespective of how radically they may differ.

1) The complex transfer function is invariant to root choice.
2) The terminating ideal transformer is invariant to the basic root pattern so that a quarter-wave matching transformer may be equally well constructed from any of the basic patterns.

The transfer function invariance is deduced through the relationship $|t(p)|^2 = 1/L(p)$, where $t(p)$ is the transfer function. Equation (5) shows that

$$t(p) = \frac{1}{P_n(p) + \sqrt{1+p^2}\,Q_{n-1}(p)}\,. \tag{33}$$

It is to be recalled that the denominator of (33) is formed of radical factors of those roots of $L(p)$, p_j, which correspond to a negative real part of $p_j/\sqrt{1+p_j^2}$. Since these roots are, in turn, the poles of $k(p)$ which remain fixed irrespective of the choice of the roots of $k(p)$, (33) is invariant to all structures having the same insertion loss function.[4]

We next show the terminating transformer invariance. With respect to Fig. 1 we find that a specification of a terminating transformer in a basic pattern implies the existence of the inverse transformer as well. The terminating transformer is found from the insertion loss for $\theta=0$ and is given through the relationship

$$4(L(0)-1)=\left(N-\frac{1}{N}\right)^2 \tag{34}$$

[4] Equation (33) is tantamount to a minimum phase statement. Since $L(p)\rightarrow\infty$ as $p\rightarrow\infty$, the transmission function $t(p)$ vanishes for $\zeta=1$ in the ζ plane. One cannot, therefore, make any direct minimum phase statements because of the nonanalyticity of ln (t) in the right-half ζ plane.

so that the transformer is specified to within an inverse. Since the insertion loss is the invariant specification to all the multiple syntheses, the transformer is an invariant to the basic root pattern. One may, therefore, always construct at least one quarter-wave transformer working into the same real impedance N^2 for each basic pattern.

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Direct Synthesis of Band-Pass Transmission Line Structures

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Abstract—Realizable band-pass (zero of transmission, i.e., infinite loss, at dc) equiripple gain functions are constructed which permit exact physical realization of systems consisting of cascaded lines and stubs. The problem of the realization of a prescribed load resistance is solved when a dc zero of transmission is present due to a shunt short-circuiting stub. The exact limits of realizable load resistance are given for equiripple band-pass gain functions and a straightforward method is presented to synthesize any desired value of load between the predetermined limits. The basis of the synthesis technique is the choice of location of the shunt stub in the cascaded chain. It is shown that the load resistance decreases monotonically as the distance of the stub from the generator increases, and it is this property which permits the realization of a wide range of load resistance from a given gain function. The method is illustrated by designs of filters, as well as a new form of broadband transformer in which the low-frequency response is suppressed by shunt stubs.

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I. Introduction

A. Application of Band-Pass Transmission Line Functions

The SYNTHESIS of cascaded, lossless, commensurate transmission line circuits is well established [1]–[11]. The results may be summarized by stating the necessary and sufficient conditions for the realizability of such a cascaded line structure [1]: Given a transmission scattering coefficient $s_{12}(j\beta l)$ [β is the propagation constant, l the line length] such that under the transformation

$$\Omega=\tan\beta l$$